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Fixed Point Theorem in Fuzzy Metric Space for Idempotent Mappings

Nidhi Verma [*] ***and Dr. Rajesh Shrivastava*** [**]
[*] *Assistant Professor, Department of Basic Science,*
Oriental Institute of Science and Technology Bhopal, (Madhya Pradesh), INDIA
[**] *Professor and Head Department of Mathematics,*
Government Science and Commerce College, Benazeer, Bhopal, (Madhya Pradesh), INDIA

(Corresponding author: Nidhi Verma)
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ABSTRACT: In this paper we generalized a common fixed point theorem in intuitionistic fuzzy metric space by using relationship between reciprocal continuity for idempotent maps in to fuzzy metric space.

Key words: fixed point, fuzzy metric space, and weak compatibility.

I. INTRODUCTION

The concept of fuzzy set was given by L.A. Zadeh [15], which laid the foundation of fuzzy mathematics. Later on, the concept of fuzzy metric space was introduced by Kramosil and Michalek [8] which is modified by George and Veeramani [4]. Also Grabiec [5] proved some fixed point results for fuzzy metric space which was developed extensively by many authors and used in various fields. Jungck [6] introduced the concept of compatible maps for a pair of self maps in 1986. Sessa [12] introduced the tradition of improving commutative condition in fixed point theorems by introducing the notion of weak commuting property. In 2006 the concept of weakly compatible maps is given by Jungck and Rhodes [7] which is more generalized than compatible maps after that R weak commutatibity of mappings of fuzzy metric space is defined by Vasuki [14] and he also proved the fuzzy version of pant's [11] theorem. Grabiec [5] gave the fuzzy version of Banach Contraction principle following Kramosil and Michalek. In 2000, B. Singh and M.S. Chouhan introduced the concept of compatibility in fuzzy metric space.
Here we will prove a fixed point theorem in fuzzy metric space by defining weak commuting in fuzzy metric space and reciprocal continuity for idempotent maps in fuzzy metric space. Our result generalize the result of M.S. Chauhan [3] and many others.

II. PRELIMINARIES

Definition 2.1 [13] A binary operation $*: [0,1] \times [0,1] \to [0,1]$is a continuous t norm if it satisfies the following condition
(a) $*$ is commutative and associative;
(b) $*$ is continuous ;
(c) $a * 1 = a \ \forall \ a\epsilon[0,1]$
(d) $a_1 * b_1 \leq a_2 * b_2$ Whenever $a_1 \leq a_2$, $b_1 \leq b_2$
Definition 2.2 [1] A 3 tuple $(X, M,*)$ is called fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set in $X^2 \times [0,\infty)$ satisfying the following conditions for all $x, y, z \in X$ and $s, t > 0$
(a) $M(x, y, 0) > 0$
(b) $M(x, y, t) = 1 \; and \; t > 0$ iff $x = y$
(c) $M(x, y, t) = M(y, x, t)$
(d) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$
(e) $M(x , y, .) : [0, \infty] \to [0,1]$ is continuous.
(f) $\lim_{t\to\infty} M(x, y, t) = 1.$
Definition 2.3 [8] A sequence $\{x_n\}$ in a fuzzy metric space $(X, M,*)$ is called Cauchy sequence if $\lim_{n\to\infty} M\,(x_{n+m}, x_n, t) = 1$ for every $t > 0$ and $m > 0$.
Definition 2.4 [8] A sequence $\{x_n\}$ in a fuzzy metric space $(X, M,*)$ is said to be convergent to $x \in X$ if $\lim_{n\to\infty} M\,(x_n, x, t) = 1$ for each $t > 0$.

Definition 2.5 [8] A fuzzy metric space $(X, M,*)$ is said to be complete if every Cauchy sequence in X converges to X.

Definition 2.6. [15] Two self mappings f and g of a fuzzy metric space $(X, M,*)$ is called weakly commuting if $M(f^2g^2x, g^2f^2x, t) \geq M(f^2x, g^2x, t)$.

Remark 2.7. [15] Weak commutative reduced to weak commuting pair (f, g) i.e.

$$M(f^2g^2x, g^2f^2x, t) \geq M(f^2gx, g^2fx, t) \geq M(fg^2x, gf^2x, t) \geq M(fgx, gfx, t) \quad \geq M(f^2x, g^2x, t)$$

If f and g are idempotent maps *i.e.* $f^2 = f$ and $g^2 = g$.

Definition 2.8 [15] Two mappings f and g of a fuzzy metric space $(X, M,*)$ into itself which are idempotent maps i.e. $f^2 = f$ and $g^2 = g$ are called reciprocally continuous on X if
$\lim_{n\to\infty} f^2g^2x_n = f^2x$ and $\lim_{n\to\infty} g^2f^2x_n = g^2x$ whenever $\{x_n\}$ is a sequence in X such that
$\lim_{n\to\infty} f^2x_n = \lim_{n\to\infty} g^2x_n = x$ for some x in X i.e.

$$M(f^2g^2x_n, g^2f^2x_n, t) \geq M(f^2gx_n, g^2fx_n, t) \geq M(fg^2x_n, gf^2x_n, t) \geq M(fgx_n, gfx_n, t) \quad \geq M(f^2x_n, g^2x_n, t) \geq M(f^2x, g^2x, t)$$

whenever $\{x_n\}$ is a sequence in X such that
$\lim_{n\to\infty} M(f^2x_n, g^2x_n, t) \geq M(f^2x, g^2x, t)\ \forall\ t > 0$ thus if two self mappings are weak commuting then they are reciprocally continuous as well.

Lemma 2.9 [15] Let $\{y_n\}$ be a sequence in fuzzy metric space $(X, M,*)$ with the condition $\lim_{t\to\infty} M(x, y, t) = 1$ and if there exist a number $k \in (0,1)$ such that $(y_{2n+2}, y_{n+1}, kt) \geq M(y_{2n+1}, y_n, t)\ \forall\ t > 0$, then $\{y_n\}$ is a Cauchy sequence in X.

Lemma 2.10 [15] Let f and g be two mappings on a complete fuzzy metric space $(X, M,*)$ into itself such that for some $k \in (0,1)$
$M(fx, gx, kt) \ \geq \min\{\ M(x, y, t), M(fx, x, t)\} \quad \forall\ x, y \in X\ and\ \forall\ t > 0.$
Then f and g have a unique common fixed point in X.

III. MAIN RESULT

Let $(X, M,*)$ be a complete fuzzy metric space and let P and Q be continuous mappings of X in X. Let A and B be two self mappings of X satisfying $[A, P]$ and $[B, Q]$ are weak commuting and
(i) $A(X) \subseteq P(X), B(X) \subseteq Q(X)$.
$(ii) M(A^2x, B^2y, t)$
$\geq\ r[\min\{\ M(P^2x, Q^2y, t), M(P^2x, A^2x, t)M(P^2x, B^2y, t), M(Q^2y, B^2y, t), M(A^2x, Q^2x, t), M(P^2y, B^2y, t)\}]$
For all $x, y \in X$, where $r: [0,1] \to [0,1]$ is continuous function such that $r(t) > t$ for each $0 \leq t \leq 1$ and $r(1) = 1$. The sequence $\{x_n\}$ and $\{y_n\}$ in X are such that $x_n \to x$ and $y_n \to y \Rightarrow\ M(x_n, y_n, t) \to M(x, y, t$, where $t > 0$ then A, B, P, Q have a unique common fixed point in X.

Proof: We define sequence $\{x_n\}$ and $\{y_n\}$ such that $y_{2n} = A^2x_{2n} = P^2x_{2n+1}$ and
$y_{2n+1} = B^2x_{2n+1} = Q^2x_{2n+2}$ for n=1,2,3,……….now we shall prove that $\{y_n\}$ is a Cauchy sequence .
Let $M_{2n} = M(y_{2n+1}, y_{2n}, t) = M(A^2x_{2n+1}, B^2x_{2n}, t)$

$$\geq r\left[\min\left\{\begin{matrix} M(P^2x_{2n+1}, Q^2x_{2n}, t), M(P^2x_{2n+1}, A^2x_{2n+1}, t)M(P^2x_{2n+1}, B^2x_{2n}, t), M(Q^2x_{2n}, B^2x_{2n}, t), \\ M(A^2x_{2n+1}, Q^2x_{2n+1}, t), M(P^2x_{2n}, B^2x_{2n}, t) \end{matrix}\right\}\right]$$

$$\geq r\left[\min\left\{\begin{matrix} M(y_{2n}, y_{2n}, t), M(y_{2n}, y_{2n+1}, t)M(y_{2n}, y_{2n}, t), M(y_{2n-1}, y_{2n}, t), \\ M(y_{2n+1}, y_{2n}, t), M(y_{2n-1}, y_{2n}, t) \end{matrix}\right\}\right]$$

$$= r\left[\min\left\{1, M_{2n}1, M_{2n-1}, M_{2n}, M_{2n-1}\right\}\right] \quad \text{…(1)}$$

If $M_{2n-1} \geq M_{2n}$ then $M_{2n} \geq r[M_{2n-1}] > M_{2n-1}$
which is a contradiction therefore $\quad M_{2n-1} \leq M_{2n}$.
From (1) we have $M_{2n} \geq r[M_{2n-1}] > M_{2n-1}$ ...(2)
Thus $\{M_{2n}: n \geq 0\}$ is increasing sequence of positive real numbers in $[0,1]$ and therefore approaches to $l_1 \geq 1$ it is clear that $l_1 = 1$ because if $l_1 < 1$ then on taking limit as $n \to \infty$ in (2) we get $l_1 \geq r[l_1] > l_1$ a contradiction hence $l_1 = 1$.

Now for any integer m

$$M(y_n, y_{n+m}, t) \geq M\left(y_n, y_{n+m}, \frac{t}{m}\right) * \ldots\ldots\ldots * M(y_{n+m-1}, y_{n+m}, \frac{t}{m})$$
$$\geq M\left(y_n, y_{n+1}, \frac{t}{m}\right) * \ldots\ldots\ldots * M\left(y_n, y_{n+1}, \frac{t}{m}\right).$$

Therefore, $\lim_{n\to\infty} M(y_n, y_{n+m}, t) \geq 1 * 1 * \ldots * 1$ because $\lim_{n\to\infty} M(y_n, y_{n+1}, t) = 1$ for $t > 0$.
Thus $\{y_n\}$ is a Cauchy sequence and by the completeness of X. $\{y_n\}$ converges to $z \in X$ so its subsequence $\{A^2x_{2n+1}\}, \{B^2x_{2n}\}\{P^2x_{2n+1}\}Q^2x_{2n}\}$ also converges to same point z.
Since $[A, P]$ is weak commuting so
$M(A^2P^2x_{2n+1}, P^2A^2x_{2n+1}, t) \geq M(A^2x_{2n+1}, P^2x_{2n}, t)$
On taking limit $n \to \infty$
$A^2P^2x_{2n+1} = P^2A^2x_{2n+1} = P^2z.$
Now we will show that $P^2z = z$.
First suppose that $P^2z \neq z$ then $\exists\ t > 0$ such that $M(P^2z, z, \mathrm{t}) < 1$
Now
$M(A^2P^2x_{2n+1}, B^2x_{2n}, t) \geq$

$$r\left[\min\left\{\begin{matrix} M(P^3x_{2n+1}, Q^2x_{2n}, t), M(P^3x_{2n+1}, A^2P^2x_{2n+1}, t)M(P^3x_{2n+1}, B^2x_{2n}, t), M(Q^2x_{2n}, B^2x_{2n}, t), \\ M(A^2P^2x_{2n+1}, Q^2x_{2n+1}, t), M(P^2x_{2n}, B^2x_{2n}, t) \end{matrix}\right\}\right]$$

$\Rightarrow$
$M(P^2z, z, t)$
$\geq r[\min\{M(P^2z, z, t), M(P^2z, P^2z, t)M(P^2z, z, t), M(z, z, t), M(P^2z, z, t), M(z, z, t)\}]$

$\Rightarrow M(P^2z, z, t) \geq r[M(P^2z, z, t)] > M(P^2z, z, t).$
Which is a contradiction. Therefore, $P^2z = z$.
Thus z is a fixed point of P. Similarly we can prove that z is also a fixed point of A *.i.e.* $A^2z = z$.
Now to prove that z is a fixed point of Q suppose z is not a fixed point of Q then for any $t > 0$,
$M(z, T^2z, t) < 1$. Now
$M(A^2z, B^2Q^2x_{2n}, t) \geq$

$$r\left[\min\left\{\begin{matrix} M(P^2z, Q^3x_{2n}, t), M(P^2z, A^2z, t)M(P^2z, B^2Q^2x_{2n}, t), M(Q^3x_{2n}, B^2Q^2x_{2n}, t), \\ M(A^2z, Q^2z, t), M(P^2Q^2x_{2n}, B^2Q^2x_{2n}, t) \end{matrix}\right\}\right]$$

$$M(z, Q^2z, t) \geq r\left[\min\left\{\begin{matrix} M(z, Q^2z, t), M(z, z, t)M(z, Q^2z, t), M(Q^2z, Q^2z, t), \\ M(z, Q^2z, t), M(Q^2z, Q^2z, t) \end{matrix}\right\}\right]$$

$\Rightarrow M(z, Q^2z, t) \geq r[M(z, Q^2z, t)] \geq M(z, Q^2z, t).$
Which is contradiction. Therefore $Q^2z = z$.
So z is a fixed point of Q. Similarly we can show that z is a common fixed point of A, B, P, Q.
Uniqueness: Suppose another fixed point $z \neq w$
then

$$M(A^2z, B^2w, t) \geq r\left[\min\left\{\begin{matrix} M(P^2z, Q^2w, t), M(P^2z, A^2w, t)M(P^2z, B^2w, t), M(Q^2w, B^2w, t), \\ M(A^2z, Q^2z, t), M(P^2w, B^2w, t) \end{matrix}\right\}\right]$$

$\Rightarrow M(z, w, t) \geq r[\min\{M(z, w, t), M(z, w, t)M(z, w, t), M(w, w, t), M(z, z, t), M(w, w, t)\}]$

$\Rightarrow M(z, w, t) \geq r[M(z, w, t)]$
Which is contradiction so $z = w$.
Hence A, B, P, Q have unique common fixed point.

REFERENCES

[1]. Banach S., (1932). Theories, lies operations, laniaries Manograie Mathematyezene, Warsaw Poland.
[2]. Chauhan, M.S. *et.al.* (2010). Common Fixed Point of Semi Compatible Maps in Fuzzy Metric Spaces, *Kathmandu University Journal of Science, Engineering and Technology,* Vol. **6**(1): 70-78.
[3]. Chauhan M.S. *et. al,* (2011). Fixed point theorem in Intuitionistic fuzzy-3 metric space using weak** compatibility, *Int. Journal of Engineering and Technology,* **3**(2): 144-150.

[4]. George, A. and Veeramani, P. (1994). On some results in fuzzy metric spaces, *Fuzzy Sets and Systems*, **64**, 395-399.
[5]. Grabiec, M. (1988). Fixed point in Fuzzy metric Spaces, *Fuzzy sets & systems,* **27**, 245-252.
[6]. Jungck, G. (1986). Compatible mappings and common fixed points, *Int. J. Math. & Math. Sci.* **9**: 771-779.
[7]. Jungck, G. and Rhoades, B.E. (1998). Fixed points for set valued functions without continuity, *Indian J. Pure and Appl. Math.*, **29**(3): 227-238.
[8]. Kramosil, O. and Michalek, J., (1975). Fuzzy metric and statistical metric spaces, *Kybernetika* **11**: 326-334.
[9]. Kutukchu S., Turkoglu D. and Yildiz C., (2006). Common fixed points of compatible maps of type (β) on fuzzy metric space. *Korean Math. Soc.* **21**(1): 89-100.
[10]. Mishra, M.K. *et.al.* (2010). An application of Fixed Point Theorems in Fuzzy Metric Spaces, *International Journal of Advanced Engineering Sciences and Technologies* Vol. **1**(2): 123-129.
[11]. Pant, R. P., (1994). Common fixed points of noncommuting mappings, *J. Math. Anal. Appl.*, **188**(2) 436-440.
[12]. Sessa, S., (1986). On a weak commutativity condition in fixed points consideration *Publ. Inst. Math* (Beogard), **46**(2): 149-135.
[13]. Singhi J. *et.al,* (2010). Fixed point theorem in fuzzy metric spaces, *Int. math. forum***, 5**(30): 1473-1480.
[14]. Vasuki, R., (1999). Common fixed points for R-weakly commuting maps in fuzzy Metric spaces, *Indian J. Pure Appl. Math*. **30**, 419-423.
[15]. Zadeh, L.A., (1965). Fuzzy sets, *Inform and Control*, **8**, 338-353.